Boat hire

Olaf is spending the day at a lake.
He wants to hire a rowing boat for some of the time.

Freya's Boat Hire charges £5 per hour.
Polly's Boat Hire charges £10 plus £1 per hour.

Whose boat should Olaf choose?

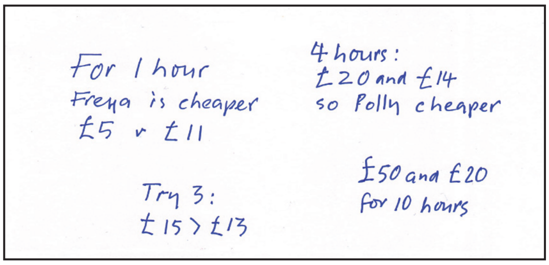
Summary

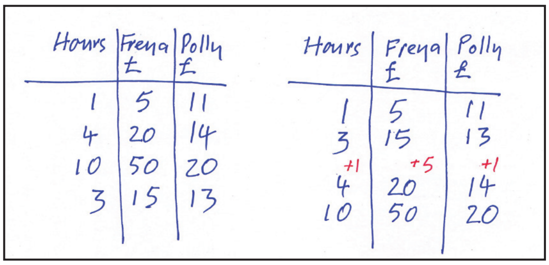
In this lesson, the boat hire problem is used to explore the two algebraic relationships underlying Freya's and Polly's different hire charges. A variety of representations are used to express the relationships:

• everyday language • algebraic expressions • tables of values • points on a Cartesian graph.

Outline of the lesson

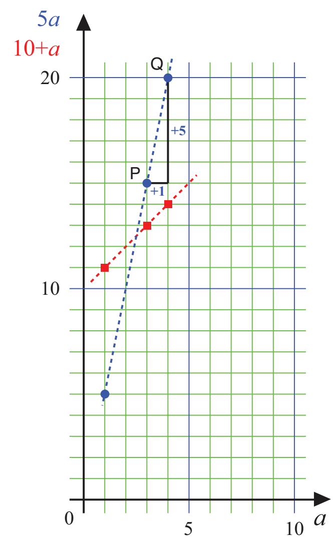
1. Display the *Boat hire* problem.
   - Ask students for their immediate responses.
   - Ask students to consider the problem further in small groups.
   - Collect numerical data on the total cost for various numbers of hours (and listen to students' arguments and conclusions - but don't pursue these at this stage).
   - Represent the data
     - 'randomly' on the board
     - in a (randomly ordered) table
     - in an ordered table (try to prompt the need for this, rather than simply produce such a table).





2. Ask students to represent the hire-rules algebraically.
   - For example, as expressions, eg $5a$ and $10 + a$
   - Or as relations, eg $b = 5a$, $b = 10 + a$.

3. Ask students to represent the data as points on a standard (Cartesian) graph.
   - Ask questions like "Are Freya's and Polly's charges ever equal?"



4. Discuss the links between the various representations and between them and the story.

Overview

Mathematical ideas

When we work with equations, we often think of a letter as representing a single number as yet unknown. Here, we are working with relations between two variables (the number of hours and the charge) and we think of the letters as representing a set of numbers.

A Cartesian graph is a particularly powerful way of representing variables, since it allows us to represent a range of values simultaneously.

Students' mathematical experiences

Students might discover some of the following

- for some values of a, Freya's hire charge ($5a$) is larger than Polly's ($10+a$), but for others it is smaller
- when $a = 2.5$ the expressions are equal
- if a increases by 1, then $5a$ increases by 5, but $10+a$ only increases by 1

  each set of points on the graph forms a pattern: each lies on a straight line.

Students might discuss

- different slopes and how these relate to the hourly charges
- continuity, ie whether some or all points on the line fit the relationship.

Some students may want to change the scales of the axes. Discourage them from doing so. The expressions have been deliberately chosen to be represented on an equally-scaled graph.

Key questions

When is Freya (or Polly) cheaper?

How could we record this more systematically?

What happens to the cost as the number of hours increases?

What if Olaf hired a boat for 1.5 hours?

Assessment and feedback

Be flexible over the organisation and timing of the lesson. Some teachers have taught this lesson over two periods.

Choose some students to contribute to a subsequent discussion. Some (less confident?) students' contributions may be more coherent if you "rehearse" with them beforehand: "That's a great idea. I'm going to ask you to explain that to the class. Let's have a go at preparing what you'll say."

Allow students time to generate algebraic expressions, but if they really struggle you may want to provide the expressions for them.

Some students may have difficulties constructing a Cartesian graph. Observe the students and decide whether you need to spend some time with either a group or the whole class teaching these skills.

You may need to prompt or challenge some students to consider whether the charges can ever be equal.

Adapting the lesson

You might want to adopt a different context for a subsequent lesson. For example, the price of an ice cream cone for different numbers of scoops, or the yearly cost of belonging to a swimming club (based on a membership fee and a cost for each visit). Choose the numbers carefully - for example, keep the multipliers small if you want to graph the relationhips, and think about where the values coincide - do you want this to occur for a simple whole number (4 ice cream scoops, say), or something more obscure (3.4 scoops, say)?

Outline of the lesson (annotated)

1. Display the *Boat hire* problem.
   - Ask students for their immediate responses.

   Students will tend to say that one or other hire firm is cheaper. Let them voice their opinions but don't discuss their reasons at this stage. (Their decisions may be based on one or two numerical examples, or they may have focussed on the lower hourly rate, in the case of Polly, or the absence of a down-payment in the case of Freya.)

   The differences of opinion should make the task more engaging for students.

   - Ask students to consider the problem further in small groups.

   This allows students to think more carefully about their and other students' ideas.

   - Collect numerical data on the total cost for various numbers of hours (and listen to students' arguments and conclusions - but don't pursue these at this stage).
   - Represent the data
     - 'randomly' on the board
     - in a (randomly ordered) table

   With the students' help, start to organise the data in a more systematic way, culminating in ordered tables.

   Some students might already have developed a coherent understanding of the problem, involving the idea that the choice of hire company depends on how long one wants the boat for. But, keep it simple and allow students time to consider how and why the two expressions differ.

     - in an ordered table (try to prompt the need for this, rather than simply produce such a table).

   Ordered tables allow one to bring out the fact that increasing the hire time by one hour, say, increases the hire charge by £5 (Freya) or £1 (Polly). This can of course be related back to the original story, but also later in the lesson to the algebraic and graphical representations of the relationship.

2. Ask students to represent the hire-rules algebraically.
   - For example, as expressions, eg $5a$ and $10 + a$
   - Or as relations, eg $b = 5a$, $b = 10 + a$.

   If students find this difficult, then present the expressions to them, without too much adoo. Or state the expressions in words, eg "5 × number of hours".

3. Ask students to represent the data as points on a standard (Cartesian) graph.
   - Ask questions like "Are Freya's and Polly's charges ever equal?"

   It is more illuminating to use axes with the same scale, even though the resulting line for Freya is then very steep. To avoid getting bogged down in the issues of choosing and drawing suitable axes (which are of course very important), you might want to give students a blank version of the graph with the axes already drawn and numbered.

4. Discuss the links between the various representations and between them and the story.

Background

Students develop their understanding of 'variable'

The kind of variable we are learning about (be it called '*a*' or 'the number of hours') has these properties:

- it is a number;
- it can take on lots of different values;
- as it changes in a systematic way, the 'dependent variable' (in this case 'the total hire charge') may also change in a systematic way.

Specifically, students might discover that

- for 'largish' values, such as a=8, 5a is larger than 10+a
- for some 'small' values, like a=1, 5a is smaller than 10+a
- there is a particular value, a=2.5, where the two expressions are equal
- if a is increased by a specific amount, then 10+a increases by the same amount, whereas 5a increases by 5 times that amount.

Students develop their ability to 'read' various representations

Each of the properties of variable mentioned above can be seen in the three representations (tables, expressions, graph). For example,

- if a value of a in a table goes up by 1, then the corresponding values of 5a have a difference of 5 (see the second table, right)
- if two points representing (a, 5a) are 1 horizontal unit apart on a graph, they will be 5 vertical units apart (see points P and Q on the graph, below right)
- the two sets of points on the graph form a pattern (in this case, they lie on two straight lines), and so help students get a general sense of the relationships, ie a sense of what happens for a range of values rather than just isolated numerical cases
- the patterns (ie the straight lines) also suggest that there is a systematic relationship between a and 5a, and between a and 10+a and that any further boat-hire points that we might calculate will also lie on the corresponding line
- the two lines cross, so there is a strong incentive to find the common point [in this case, (2.5, 12.5)] and to consider what it might mean in the present boat-hire context.

Students create algebraic expressions

The problem can be solved perfectly well without algebraic symbolisation; on the other hand, the expressions do provide a convenient shorthand and the context should help make them meaningful to the students. It also allows us to refer explicitly to variables.

Students engage with the notion of continuity

The graph, in particular, prompts the question of 'intermediate' values: given that each set of points lies on a line, what about some other points on that line: do they fit the algebraic relationship, and do they fit the boat-hire story? And do 'all' points on the line fit the relationship?

Many hire firms charge in whole numbers of hours (so a boat used for 2 hours 15 mins, say, would be charged for 3 hours). This results in a step function, represented on a Cartesian graph by a set of horizontal lines.

not ordered

a	5a
1	5
4	20
3	15

ordered

	a	5a	
	1	5	
	3	15	
+1	4	20	+5